Analytic expressions for the equivalent diameters of rectangular cross section conductors

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Previously I had written up some notes where I calculated the equivalent resistance of a rectangular cross section conductor by dividing the surface up into segments, using a piecewise constant charge distribution over each segment, and solving for the charge distribution that gives an equipotential on the surface. I then calculated an equivalent diameter for the r.f. resistance of this conductor. It is also easy to calculate the equivalent diameter of a circular cross section conductor that gives the same capacitance per unit length. Since the capacitance is calculated as the ratio of the total charge to the potential difference. This equivalent diameter is the one used in replacing the rectangular cross section element with a circular cross section one for perfectly conducting approximation in antenna codes, while the resistive diameter can be used to evaluate the resistive loss terms.

Y.T. Lo, "A note on the cylindrical antenna of noncircular cross section," J. App. Phys. 24, 1338 (1953), solved for the self impedance equivalent diameter of regular polygon cross section conductors. Here I extend his analysis to rectangular cross section conductors. I calculate both the equivalent self impedance diameter and the equivalent r.f. resistance diameter.

Lo used the conformal mapping from the u to z planes given by

$$dz = \prod_{i=1}^{n} (u - a_i)^{\alpha_i} u^{-2} du, \tag{1}$$

and showed how it can map the exterior of a circle to the exterior of an n-sided regular polygon if the a_i are the n nth roots of 1, and α_i are $2/n$. For rectangles, we keep $\alpha_i = 2/n$ and keep the the a_i on the unit circle, but not evenly spaced. Specifically, we take angles θ, π, $\pi + \theta$, 2π, to make a rectangle. The value of θ determines the two sides of the rectangle. $\theta = \pi/2$ gives a square and reduces to Lo's solution for n=4. The function Eq. 1 can now be written as

$$dz = [u^2 e^{-i\theta} + u^{-2} e^{i\theta} - 2\cos(\theta)]^{\frac{1}{2}} u^{-1} du \tag{2}$$

where we drop an unimportant constant phase factor of $\exp(-i\theta/2)$. The length of the two sides of the rectangle corresponding to the original unit circle are given by integrating u around the unit circle, That is with $u = \rho e^{i\phi}$, with $\rho = 1$,

$$dz = [2\cos(2\phi - \theta) - 2\cos(\theta)]^{\frac{1}{2}} d\phi \tag{3}$$

Choosing θ less than or equal to $\pi/2$, the lengths of the two sides of the rectangle corresponding to the unit circle are

$$s_1 = \int_0^\theta \left| \frac{dz}{d\phi} \right| d\phi$$

$$
\begin{aligned}
&= \sqrt{2}\int_0^{\theta} |\cos(2\phi-\theta)-\cos(\theta)|^{\frac{1}{2}} d\phi \\
s_2 &= \int_\theta^{\pi} \left|\frac{dz}{d\phi}\right| d\phi \\
&= \sqrt{2}\int_\theta^{\pi} |\cos(2\phi-\theta)-\cos(\theta)|^{\frac{1}{2}} d\phi \qquad (4)
\end{aligned}
$$

with s_2 the longer side. The form of these integrals is very similar to the form we get when we solve for the exact dynamics of a simple pendulum. The integrals can be written in terms of complete elliptic integrals with the standard substitution

$$
\sin(\phi-\theta/2) = \sin(\theta/2)\sin(\gamma) \qquad (5)
$$

to be

$$
\begin{aligned}
s_1 &= 4[E(k) - k'^2\ K(k)] \\
s_2 &= 4[E(k') - k^2\ K(k')] \qquad (6)
\end{aligned}
$$

where $k = \sin(\theta/2)$, and $k' = \sqrt{1-k^2}$ is the complementary modulus. $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds defined as usual as

$$
\begin{aligned}
K(k) &= \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2(\gamma)}} d\gamma \\
E(k) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2(\gamma)} d\gamma \qquad (7)
\end{aligned}
$$

The effective self impedance diameter for this size rectangle is two. The effective self impedance diameter of a rectangular cross section conductor of width w and thickness t is given by

$$
d_{self} = \frac{2w}{s_2} = w\ \frac{1}{2[E(k') - k^2\ K(k')]} \qquad (8)
$$

where θ is chosen so that s_2/s_1 equals w/t, that is

$$
\frac{w}{t} = \frac{E(k') - k^2\ K(k')}{E(k) - k'^2\ K(k)} \qquad (9)
$$

The charge density can be obtained similarly. The charge density is proportional to the perpendicular electric field. The equipotentials for the u system are circles, so we can calculate other equipotentials by choosing $\rho > 1$. We can take the normal derivative of the potential to get the perpendicular electric field. You can think of this as evaluating the distance h that a nearby equipotential is from the rectangle. The derivative of the potential is then just the difference in the potentials divided by this distance. Taking the derivative with respect to ρ and calculating this distance gives

$$
\begin{aligned}
E(\phi) &= \left[2^{\frac{3}{2}}\int_0^{\phi} \frac{\sin(2\phi'-\theta)}{|\cos(2\phi'-\theta)-\cos(\theta)|^{\frac{1}{2}}} d\phi'\right]^{-1} \\
&= 2^{-\frac{3}{2}}|\cos(2\phi-\theta)-\cos(\theta)|^{-\frac{1}{2}} \qquad (10)
\end{aligned}
$$

Normalizing this to 1, we get the normalized charge density to be

$$\sigma(\phi) = \frac{2^{-\frac{1}{2}}}{2\pi} |\cos(\theta) - \cos(2\phi - \theta)|^{-\frac{1}{2}} \tag{11}$$

The integral of the charge density squared around the square is then

$$\int_0^{2\pi} \sigma(\phi)^2 \left|\frac{dz}{d\phi}\right| d\phi = \frac{1}{\sqrt{32}\pi^2} \int_0^{\pi} |\cos(2\phi - \theta) - \cos(\theta)|^{-\frac{1}{2}} d\phi. \tag{12}$$

These integrals can also be done in terms of elliptic integrals giving

$$\int_0^{2\pi} \sigma(\phi)^2 \left|\frac{dz}{d\phi}\right| d\phi = \frac{1}{4\pi^2} [K(k) + K(k')] \tag{13}$$

and the effective resistance diameter is

$$d_{resistance} = w \, \frac{\pi}{2[K(k) + K(k')][E(k') - k^2 \; K(k')]} \tag{14}$$

Previously, I fit my brute force numerical calculations to the expressions

$$d_{self} = w[0.5 + 0.9t/w - 0.22(t/w)^2] \tag{15}$$

and

$$d_{resistance} = w \frac{1}{1 + 1.13 \log_{10}(w/t)} \tag{16}$$

Table 1: The calculated values of the self impedance effective diamter and the resistance effective diameter as a function of the cross section width divided by thickness compared to the fits. The value of k is the calculated value to get the correct w/t ratio.

w/t	k	d_{self}	Eq. 15	$d_{resistance}$	Eq. 16
1	0.70711	1.18034 w	1.18 w	1.00000 w	1.00 w
2	0.58862	0.87476 w	0.90 w	0.73203 w	0.75 w
5	0.43001	0.67185 w	0.67 w	0.53502 w	0.56 w
10	0.32480	0.59529 w	0.59 w	0.44872 w	0.47 w
20	0.23912	0.55265 w	0.54 w	0.39123 w	0.40 w
50	0.15567	0.52383 w	0.52 w	0.33997 w	0.34 w
100	0.11131	0.51299 w	0.51 w	0.31200 w	0.31 w
200	0.07920	0.50704 w	0.50 w	0.28962 w	0.28 w
500	0.05030	0.50310 w	0.50 w	0.26564 w	0.25 w
1000	0.03562	0.50166 w	0.50 w	0.25042 w	0.23 w

It seems likely to me that this calculation would have been done long ago and is probably buried somewhere in the literature, however, I have not found it anywhere. All the integrals can be evaluated in terms of elliptic integrals, and tables of these have been available for a very long time.