Approximate R.F. Resistance of Rectangular Cross Section Conductors

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1 Introduction

Small loop antennas such as the AEA-Isoloop sometimes use rectangular cross section conductors, and sometimes use circular cross section tubing for the antenna element. In these antennas, resistive losses are important in determining the efficiency. Here, I approximately calculate the resistance of rectangular cross section elements. For design purposes my results can be summarized by the equation

$$d_{\text{effective}} = \frac{w}{1 + 1.13 \log_{10}(w/t)}, \tag{1}$$

where $d_{\text{effective}}$ is the diameter of a circular cross section conductor with the same r.f. resistance, w is the width of the rectangular conductor, and t is the thickness, with t less than w.

2 Calculation Method

My approximate calculation is reasonable if:

1. The width of the strip is much much smaller than a wavelength,

2. The strip is far enough from other conductors that their fields don't disturb the distribution,

3. The strip is much thicker than a skin depth.

You can think of the calculation as solving for the current distribution on the conductors of two conductor open wire transmission line where both conductors have the same rectangular cross section. I then let the separation between the two conductors get large enough so the fields from the other conductor do not change the current distribution.

A two-wire transmission line with perfectly conducting wires operates in its TEM mode. The solution of Maxwell's equations for this mode can be calculated by solving the simpler 2-dimensional electrostatic problem for the surface charge density on the wires. The surface current distribution is identical to the charge distribution. Once I have the surface charge distribution, I then apply a very approximate perturbation of boundary conditions method. That is the tangential component of the magnetic field just outside the surface is proportional to the surface current at that point.

If the surface is planar, the current in a good but imperfect conductor dies out exponentially with distance into the conductor. The usual skin depth gives the exponential decay rate. Obviously, near the corners, the assumption of a planar solution is incorrect. However, if the dimensions of the conductor are all large compared to the skin depth, this approximation will be reasonable since the regions where it breaks down will only contribute a small amount to the effective resistance. A true perturbation of boundary conditions method could be used to get a better approximation near the corners of the conductor, but I have not done that here. Using this simple planar approximation, the resistance per unit length of a conductor is

$$\frac{|I|^2 R}{2} = \frac{\delta}{2} \oint d\vec{r} \frac{|J_z(\vec{r})|^2}{\sigma} \tag{2}$$

where

$$I = \delta \oint d\vec{r} J_z(\vec{r}) \tag{3}$$

is the current on the conductor and the integrals are around the surface cross section of the conductor, σ is the conductivity, and δ is the skin depth $\sqrt{1/(\mu \pi f \sigma)}$, and f is the frequency. Notice that if the current is uniformly distributed around the surface, that this formula reduces to $R = 1/(p\delta\sigma)$ where p is the perimeter of the conductor cross section. The effective diameter $d_{\text{effective}}$ is calculated as the diameter of the circular cross section conductor with the same resistance,

$$
\begin{aligned}
d_{\text{effective}} &= \frac{1}{\pi \delta \sigma R} \\
&= \frac{|\oint d\vec{r} J_z(\vec{r})|^2}{\pi \oint d\vec{r} |J_z(\vec{r})|^2}
\end{aligned}
\tag{4}
$$

Since J_z has the same form as the charge density, I can use the charge density in place J_z in Eq. 4

To calculate the charge density, I divided the surface up into small segments and assumed the charge density was constant over each segment. I then solved for the magnitude of the charge on each segment by requiring the average potential on each segment to be the same. In figures 2 and 2, I show the calculated charge density as a function of the distance from the edge of a square cross section conductor. The surface charge density bunches at the corners. Figure 2 displays the calculated charge density and the function $0.124x^{-1/3}$. The calculated charge density has the correct $x^{-1/3}$ divergence at the corners just like analytic theory says should be true.

3   Results for Rectangular Conductors

Applying this method to a flat strip of rectangular cross section, I calculated the effective diameter. I increased the number of basis functions until the results converged to the accuracy shown. Figure 3 shows a plot of these results along with the function given in Eq. 1.

4   Consistency Checks

To check that the code is giving sensible results, I applied it to the case of a microstrip transmission line with air insulation. In this case, the rectangular cross section conductor is suspended over a

ground plane. Originally I checked using some numbers out of Terman, Reference data for Radio Engineers. The microstrip results there are apparently from an approximate conformal mapping method of Assadourian and Rimai, Proc. IRE 40, 1651 (1952), but my values are in fairly good agreement with more modern calculations too.

The microstrip conductor has width w, thickness t, just like our previous calculations. The distance from the bottom of the strip to the ground plane is h. Terman gives the following formula for the conductor loss

$$\text{Loss}_\text{Terman} \text{ dB/foot} = 7.25 \times 10^{-5}\alpha \left[1/\text{h(inches)}\sqrt{\text{f(MHz)}}\right], \tag{5}$$

with a graph for the value of α. Using the same electrostatic method as I used for the isolated conductor I calculated the resistance of the microstrip conductors. I calculated the characteristic impedance Z_0 from the capacitance, and the ratio is proportionate to the attenuation.

Table 2 gives a comparison of values of α 7.25×10^{-5} from Terman and my calculations. The loss calculation includes loss in the ground plane copper. I also give my calculated characteristic impedance Z_0, and the effective diameter of the strip. This is the diameter of an isolated circular rod with the same r.f. resistance as just the strip conductor (not ground plane).

So mostly my code agrees with Terman within 5 to 10 percent. The Terman data are old and not readily available. More modern results can be found at the link http://www.mit.edu/~mcmahill/software/mstrip

It is a web based microstrip calculator. Change the input data on that web page to:

```
Width = 10
Length = 1000
Frequency = 30 GHz
Er = 1.0001 (1 causes a problem)
H = 10
Tmet = 1
Rho = 1
Rough = 0.0
Tand = 0.0
Physical Units = mils
```

Table 1: The calculated effective diameter of rectangular strips with various width to thickness ratios.

w/t	$d_{\text{effective}}/w$
1	1.02
2	0.74
5	0.54
10	0.46
20	0.40
50	0.34
100	0.31

Then click on the $----->$ button. This gives the calculator the input data that corresponds to $w/h = 1$, $t/h = .1$.

Inputing my other values to this calculator gives Z_0 values that agree within 0.1 ohm of mine, and reasonable agreement for the loss values. The web description claims the Z_0 better than 1 percent of the correct answer; the loss results are stated to be less accurate but still useful. Converting Terman's and my numbers from table 2 to the same units as the web calculator gives the results shown in table 3.

I then used my code and allowed the height to get large. I used a piecewise constant charge density on the conductors with 100 elements along w and 50 along t, except for the infinite height values which are the well converged values given in table 1.

For h very small the current on the bottom of the strip will be nearly uniform, and we expect d effective to go to $1/\pi = .32$ So all my results seem to be consistent with standard microstrip values. As the height increases, there are two effects. One, substantial current starts to flow on the top surface of the strip, this decreases the resistance, and two, the current "bunches" at the edges. This increases the resistance. These two effects tend to cancel.

I also did some calculations for a "stadium" geometry. This is a strip with width w and thickness t, but the edges have semicircles of diameter t so there aren't any sharp corners and the calculation converges faster. To be specific the cross section is a rectangle of width $w - t$ and thickness t, with semicircles of diameter t added to the edges. Calculating the effective diameter of the stadium geometry gives the results in table 5 which I fit to the equation,

$$d_{\text{effective}} = \frac{w}{1 + \log_{10}(w/t)} \quad \text{stadium geometry} \tag{6}$$

5   MFJ-1786 and AEA Isoloop

According to W6TNS on his Web page http://www.gate.net/~donstone/aea.html , the AEA Iso-loop has w=1.5 inches, t=.063 inches, which if you believe my numbers has an r.f. resistance equivalent to about 0.6 inch diameter tubing. The MFJ-1786 loop has a 1.05 inch diameter round conductor according to an MFJ advertisement in QST.

Table 2: A comparison of loss calculations for microstrip line from my calculations and the tabulation of Terman.

w/h	t/h	Terman	My Value	% difference	My(Z_0)	My($d_{\text{effective}}$)
10	.1	7.25 $\times 10^{-5}$	7.16 $\times 10^{-5}$	1	28.7	3.55 h
1	.1	10.73 $\times 10^{-5}$	9.46 $\times 10^{-5}$	14	118.7	0.47 h
10	.01	7.68 $\times 10^{-5}$	8.42 $\times 10^{-5}$	9	29.0	2.72 h
3	.01	9.57 $\times 10^{-5}$	9.90 $\times 10^{-5}$	3	69.5	0.82 h
2	.01	10.37 $\times 10^{-5}$	10.82 $\times 10^{-5}$	4	88.5	0.56 h
1	.01	13.78 $\times 10^{-5}$	13.18 $\times 10^{-5}$	5	125.2	0.30 h

Table 3: A Comparison of Loss calculated using the web based calculator, my calculation, and the tabulation of Terman.

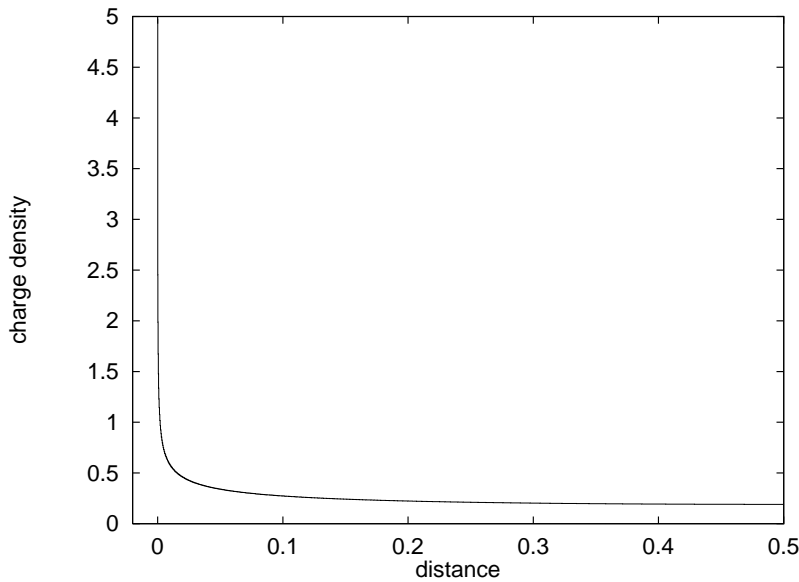| Width=w	H=h	Tmet=t	Calculator dB/inch	My dB/inch	Terman dB/inch
100	10	1	0.1	0.103	0.105
10	10	1	0.14	0.137	0.155
100	10	.1	0.11	0.122	0.111
30	10	.1	0.13	0.143	0.138
20	10	.1	0.14	0.156	0.150
10	10	.1	0.18	0.190	0.199

Figure 1: The calculated charge density on a square cross section conductor with unit length side as a function of the distance from a corner.
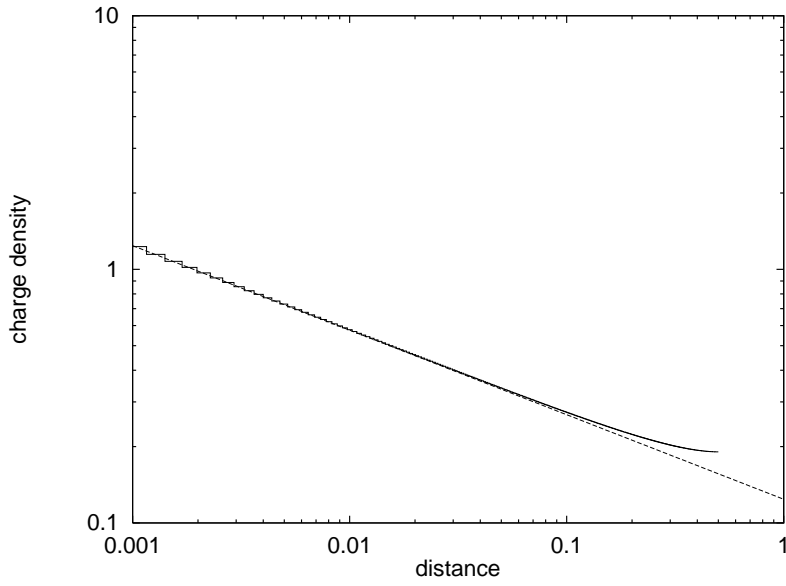
Figure 2: A log-log plot of the calculated charge density on a square cross section conductor with unit length side as a function of the distance from a corner. The dashed line is the function $0.124x^{-1/3}$.
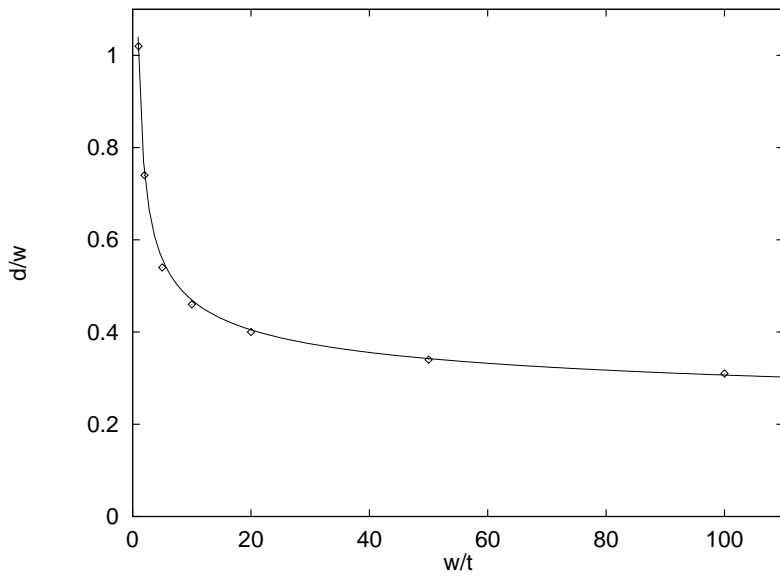

Figure 3: A plot of the calculated effective diameter of a rectangular cross section conductor. The points are the calculate values, and the curve is fit to this data and is given by Eq. 1.

Table 4: Calculated effective diameter of the strip conductor of a microstrip line as a function of its height for various thicknesses.

w	t	h	$d_{\text{effective}}$
1	1	.001	.32
1	1	.01	.37
1	1	.1	.57
1	1	1	.98
1	1	10	1.06
1	1	100	1.06
1	1	∞	1.02
1	.1	.001	.32
1	.1	.01	.35
1	.1	.1	.44
1	.1	1	.47
1	.1	10	.47
1	.1	100	.47
1	.1	∞	.46
1	.01	.001	.32
1	.01	.01	.34
1	.01	.1	.35
1	.01	1	.29
1	.01	10	.29
1	.01	100	.29
1	.01	∞	.31

Table 5: The calculated effective diameter of stadium shaped strips with various width to thickness ratios.

w/t	$d_{\text{effective}}/w$
1	1.00
2	0.77
5	0.58
10	0.49
20	0.43
50	0.37
100	0.34